

FACTA UNIVERSITATIS (NIŠ)  
 SER. MATH. INFORM. Vol. 32, No 5 (2017), 629–657  
<https://doi.org/10.22190/FUMI1705629T>

## LOCAL EXISTENCE AND SUFFICIENT CONDITIONS OF THE NON-GLOBAL SOLUTION FOR WEIGHTED DAMPED WAVE EQUATIONS \*

Hadj Kaddour Tayeb and Ali Hakem

**Abstract.** In this paper we study the following Cauchy problem of the weighted damped wave equation with nonlinear memory

$$u_{tt} - \Delta u + g(x) |u|^{m-1} u_t = \int_0^t (t - \tau)^{-\gamma} |u(\tau, \cdot)|^p d\tau$$

$$u(0, x) = u_0(x), u_t(0, x) = u_1(x), \quad x \in \mathbb{R}^N,$$

in the multi-dimensional real space  $\mathbb{R}^n$ . where,  $m > 1$ ,  $p > 1$ ,  $0 < \gamma < 1$  and  $\Delta$  is the usual Laplace operator and  $g$  is a positive smooth function which will be specified later. Firstly, we will prove the existence and uniqueness of the local solution theorem and, secondly, the nonexistence of the global solutions theorem is established.

**Keywords:** Damped wave equation; weak solution; test function; fractional derivative

### 1. Introduction

In 2008, Cazenave and al [6] generalized some results obtained by Fujita [5] in 1966 when they studied the following equation

$$(1.1) \quad \partial_t u(t, x) - \Delta u(t, x) = \int_0^t (t - \tau)^{-\gamma} |u|^{p-1} u(\tau, x) d\tau$$

where  $0 \leq \gamma < 1$  and  $u_0 \in C_0(\mathbb{R}^N)$ . Their results are the following. Let

$$p_\gamma = 1 + \frac{4 - 2\gamma}{(N - 2 + 2\gamma)_+} \text{ and } p^* = \max \left( p_\gamma, \frac{1}{\gamma} \right) \\ \text{with } (N - 2 + 2\gamma)_+ = \max(N - 2 + 2\gamma, 0),$$

then

---

Received June 21, 2016; accepted August 06, 2017

2010 *Mathematics Subject Classification.* Primary 26A33, 35B33; Secondary 74G25, 74H35.

\*The authors were supported in part by CNEPRU. COOL 03 UN 220 120 150001. ALGERIA.

1. If  $\gamma \neq 0, p \leq p^*$  and  $u_0 > 0$ , then the solution  $u$  of (1.1) blows up in finite time.
2. If  $\gamma \neq 0, p > p^*$  and  $u_0 \in L_{q^*}(\mathbb{R}^N)$  where  $q^* = \frac{(p-1)N}{4-2\gamma}$  with  $\|u_0\|_{L_{q^*}}$  small enough, then  $u$  exists globally.

In particular, they proved that the critical exponent in Fujita's sense  $p^*$  is not the one predicted by scaling. But this is not a surprising result since it is well known that scaling is efficient only for parabolic equations and not for pseudo-parabolic ones. To show this, it is sufficient to note that, formally, equation (1.1) is equivalent to

$$D_{0|t}^\alpha u_t - D_{0|t}^\alpha \Delta u = \Gamma(\alpha) |u|^{p-1} u,$$

where  $\alpha = 1 - \gamma$  and  $D_{0|t}^\alpha$  is the fractional derivative operator of order  $\alpha$  ( $\alpha \in ]0, 1[$ ) of Riemann-Liouville defined by

$$D_{0|t}^\alpha u = \frac{d}{dt} I_{0|t}^{1-\alpha} u.$$

3. In the case of  $\gamma = 0$ , Souplet [21] has showed that a non-zero positive solution blows up in finite time.

After that, precisely in 2013, M. Berbiche and A. Hakem [8] thought about generalizing the above results for problems more difficult than those they addressed in the study of the following problem

$$(1.2) \quad \partial_t^2 u(t, x) - \Delta u(t, x) + |u|^{m-1} \partial_t u(t, x) = \int_0^t (t - \tau)^{-\gamma} |u(\tau, x)|^p d\tau,$$

which describes a damped wave equation with nonlinear memory and the damping is not linear, either. Specifically, they proved that if  $p > m > 1$  and the initial datum satisfies

$$\int_{\mathbb{R}^N} u_0(x) dx > 0, \int_{\mathbb{R}^N} |u_0|^{m-1} u_0(x) dx > 0, \int_{\mathbb{R}^N} u_1(x) dx > 0,$$

and if

$$N \leq \left( \frac{2(m + (1 - \gamma)p)}{(p - 1 + (1 - \gamma)(m - 1))}, \frac{2(1 + (2 - \gamma)p)}{\left(\frac{(p-1)(2-\gamma)}{p-m} + \gamma - 1\right)(p - 1)} \right) \quad \text{or} \quad p \leq \frac{1}{\gamma},$$

then the solution of the equation (1.2) with such initial data  $u_0$  and  $u_1$  does not exist globally in time.

In this paper we would like to obtain similar results for a problem that is more general than (1.2), namely, a weighted damped wave equation with nonlinear memory, which reads

$$(1.3) \quad \begin{cases} \partial_t^2 u(t, x) - \Delta u(t, x) + g(x) |u|^{m-1} \partial_t u(t, x) = \int_0^t (t - \tau)^{-\gamma} |u(\tau, x)|^p d\tau, \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \mathbb{R}^N, \end{cases}$$

where the weight  $g(x)$  is stationary (independent of time).

Firstly, our purpose in this paper is to explore the local existence and uniqueness of the solution for the equation (1.3) by using the fixed point theorem. Secondly, we shall make full use of the test function method considered by Fino [7], Berbiche and Hakem [8], Pohozaev and Tesei [15], Mitidieri and Pohozaev ([16], [17]) and by Zhang [12] to prove blow-up results of the solution to the equation (1.3).

**Remark 1.1.** Throughout this paper, the constants will be denoted  $C$  and are different from one place to another.

## 2. Notations and Preliminary results

For a multi-index  $\alpha = (\alpha_1, \alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$  we denote by

1.  $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_N$  the length of the multi-index  $\alpha$ .
2.  $\alpha!$  for the factorial of  $\alpha$ :  $\alpha! = \alpha_1! \alpha_2! \dots \alpha_N!$
3. For all  $\beta = (\beta_1, \dots, \beta_N) \in \mathbb{N}^N$ ,  $x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N$ , we define

$$D_x^\beta \varphi = \frac{\partial_x^{|\beta|} \varphi}{\partial_{x_1}^{\beta_1} \dots \partial_{x_N}^{\beta_N}}, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

where  $\mathcal{D}(\Omega)$  is the set of  $\mathcal{C}^\infty$  functions with compact supports included in  $\Omega$ .

We denote by  $H^s(\mathbb{R}^N)$  the Sobolev space defined by

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N), \quad \frac{\partial u}{\partial x_i} \in L^2(\mathbb{R}^N), \quad i = 1, 2, \dots, N \right\},$$

where the derivation is considered in the distribution sense. We also need some results as Sobolev's embedding theorems. We need the following Lemmas.

**Lemma 2.1.** ([4]) If  $s > N/2$  then one has

$$H^s(\mathbb{R}^N) \subset \mathcal{C}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N),$$

where the inclusion is continuous. Indeed, there exists a constant  $C > 0$  such that

$$\forall u \in H^s(\mathbb{R}^N), \quad \|u\|_{L^\infty(\mathbb{R}^N)} \leq C \|u\|_{H^s(\mathbb{R}^N)}.$$

**Lemma 2.2.** ([8]) Assume that  $s_1, s_2 \geq s > N/2$ , then for all  $u \in H^{s_1}(\mathbb{R}^N)$  and  $v \in H^{s_2}(\mathbb{R}^N)$  there exists a positive constant  $C$  independent of  $u$  and  $v$  such that

$$\|uv\|_{H^s(\mathbb{R}^N)} \leq C \|u\|_{H^{s_1}(\mathbb{R}^N)} \|v\|_{H^{s_2}(\mathbb{R}^N)}.$$

**Lemma 2.3.** ([8]) Let  $s, N \geq 1$  such that  $N \geq s - 1$ , then for all nonnegative functions  $u$  belonging to  $L^\infty(\mathbb{R}^N) \cap H^{s-1}(\mathbb{R}^N)$  and  $n \in \mathbb{N}^*$  we have

1.  $u^n \in H^{s-1}(\mathbb{R}^N)$ .
2. There exists a constant  $C > 0$  such that

$$\|u^n\|_{H^{s-1}(\mathbb{R}^N)} \leq C \|u\|_{L^\infty(\mathbb{R}^N)}^{n-1} \|u\|_{H^{s-1}(\mathbb{R}^N)}.$$

The next lemma is an immediate consequence of (proposition 3.7 pp. 11 in [4]). Invoking the fact that  $f$  is bounded with all its derivatives as it is mentioned in Lemma 2.4 and by using the below Leibnitz formula (see formula 3.23 p.11 in [4]),

$$(2.1) \quad D^\alpha(fu) = \sum_{\beta+\gamma=\alpha} C_\alpha^\beta (D^\beta f)(D^\gamma u), \quad \text{where } C_\alpha^\beta = \frac{\alpha!}{\beta!(\alpha-\beta)!}$$

**Lemma 2.4.** *Let  $s \in \mathbb{N}^*$ ,  $u \in H^{s-1}(\mathbb{R}^N)$  and  $f$  be a real valued bounded function with all its derivatives, then one has*

1.  $fu \in H^{s-1}(\mathbb{R}^N)$ .
2. Denoting  $C_f = \sup_{x \in \mathbb{R}^N} \left( \sup_{|\alpha| \leq s-1} |D_x^\alpha f(x)| \right)$  then  $\|fu\|_{H^{s-1}(\mathbb{R}^N)} \leq C_f \|u\|_{H^{s-1}(\mathbb{R}^N)}$ .

### 3. Well-posedness of the problem

#### 3.1. Introduction and statement of our problem

In this section we will prove the theorem of the existence and uniqueness of solutions to the following Cauchy problem, which described a weighted damped wave equation with nonlinear memory

$$(P) \begin{cases} u_{tt} - \Delta u + g(x)|u|^{m-1}u_t = \int_0^t (t-\tau)^{-\gamma} |u|^p d\tau \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n \\ u_t(0, x) = u_1(x), & x \in \mathbb{R}^n, \end{cases}$$

with  $m > 1$ ,  $p > 1$ ,  $0 < \gamma < 1$ ,  $\Delta$  is the usual Laplace operator and  $g$  is a function which will be specified later.

#### 3.2. Main result

Now we are able to state the main result and its proof concerning the local existence and uniqueness of the solution to *problem (P)*.

**Theorem 3.1.** *Let  $N \geq 1, s > \frac{N}{2}, m, p \in (1, \infty)$  such that  $m, p > s - 1$ . Assume that  $g \in C^s(\mathbb{R}^N)$  such that  $g$  is positive and satisfies*

$$\forall \beta = (\beta_1, \dots, \beta_N) \in \mathbb{N}^N, \quad |\beta| \leq s \text{ such that: } D_x^\beta g = \frac{\partial^{|\beta|} g}{\partial x_1^{\beta_1} \dots \partial x_N^{\beta_N}} = \mathcal{O}(1),$$

*uniformly with respect to  $x \in \mathbb{R}^N$ , then for any  $(u_0, u_1) \in H^s(\mathbb{R}^N) \times H^{s-1}(\mathbb{R}^N)$ , the problem (P) admits a unique solution*

$$u \in \mathcal{C}([0, T], H^s(\mathbb{R}^N)) \cap \mathcal{C}^1([0, T], H^{s-1}(\mathbb{R}^N)).$$

Where  $T$  depends only on  $\|u_0\|_{H^s(\mathbb{R}^N)} + \|u_1\|_{H^{s-1}(\mathbb{R}^N)}$ .

The main tool for the proof of *Theorem 3.1* is the fixed point theorem. For this reason, we need a suitable functional space and a contraction mapping. To do this, we define for some  $T > 0$  and  $M > 0$  the following functional spaces.

$$X_T = \mathcal{C}([0, T], H^s(\mathbb{R}^N)) \cap \mathcal{C}^1([0, T], H^{s-1}(\mathbb{R}^N)),$$

$$E_T = L^\infty([0, T], H^s(\mathbb{R}^N)) \cap W^{1,\infty}([0, T], H^{s-1}(\mathbb{R}^N)),$$

$$E_{T,M} = \left\{ u \in E_T; \sup_{t \in [0, T]} \left( \|u\|_{H^s(\mathbb{R}^N)} + \|u_t\|_{H^{s-1}(\mathbb{R}^N)} \right) \leq M \right\},$$

and we put by definition

$$X_{T,M} := X_T \cap E_{T,M}.$$

**Remark 3.1.** It is easy to remark that one has  $X_T \subset E_T$  and  $X_{T,M} \subset E_{T,M}$  for all  $T > 0$  and  $M > 0$ .

Still denoting

$$\begin{aligned} P_\alpha(g, \partial_t)u &= -g(x)|u|^{m-1}u_t + \int_0^t (t-\tau)^{-\gamma} |u(\tau, \cdot)|^p d\tau \\ (3.1) \quad &= -g(x)|u|^{m-1}u_t + \Gamma(\alpha) I_{0|t}^\alpha (|u|^p), \end{aligned}$$

where  $\alpha = 1 - \gamma$  and  $I_{0|t}^\alpha$  is the Riemann-Liouville fractional integral of order  $\alpha$  ( $\alpha \in ]0, 1[$ ) defined by (See [18])

$$(3.2) \quad I_{0|t}^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(\tau)}{(t-\tau)^{1-\alpha}} d\tau,$$

and  $\Gamma$  is the usual Euler's Gamma function. First of all, we will state some results to prove *Theorem 3.1*.

**Lemma 3.1.** *Let  $P_\alpha(g, \partial_t)$  be the operator defined by (3.1), then for all  $t \in [0, T]$  and  $u \in E_T$  one has*

$$\begin{aligned} \int_0^t P_\alpha(g, \partial_t)u(\tau, \cdot) d\tau &= \frac{g(x)}{m} \left( |u(0, \cdot)|^{m-1} u(0, \cdot) - |u(t, \cdot)|^{m-1} u(t, \cdot) \right) \\ &+ \frac{1}{\alpha} \int_0^t (t - \tau)^\alpha |u(\tau, \cdot)|^p d\tau. \end{aligned}$$

*Proof.* We have

$$\begin{aligned} (3.3) \quad \int_0^t P_\alpha(g, \partial_t)u(\tau, \cdot) d\tau &= -g(x) \int_0^t |u(\tau, \cdot)|^{m-1} u_t(\tau, \cdot) d\tau \\ &+ \int_0^t \int_0^\tau (\tau - s)^{\alpha-1} |u(s, \cdot)|^p ds d\tau = -g(x)I + J. \end{aligned}$$

For  $I$ , it is enough to note that for all  $u \in E_T$ , we have

$$(3.4) \quad \partial_t(|u(t, \cdot)|^{m-1} u(t, \cdot)) = m |u(t, \cdot)|^{m-1} u_t(t, \cdot),$$

and for  $J$  we just use Fubini's theorem (Theorem 1.1.7, pp. 8 in [3]) to calculate the integral with respect to  $\tau$  and find

$$(3.5) \quad J = \frac{1}{\alpha} \int_0^t (t - s)^\alpha |u(s, \cdot)|^p ds.$$

Combining (3.4) and (3.5) into (3.3) we get

$$\begin{aligned} \int_0^t P_\alpha(g, \partial_t)u(\tau, \cdot) d\tau &= -\frac{1}{m} g(\cdot) \left( |u(t, \cdot)|^{m-1} u(t, \cdot) - |u(0, \cdot)|^{m-1} u(0, \cdot) \right) \\ &+ \frac{1}{\alpha} \int_0^t (t - \tau)^\alpha |u(\tau, \cdot)|^p d\tau. \end{aligned}$$

This completes the proof of Lemma 3.1.  $\square$

Thanks to Lemma 3.1 for estimating the norm in  $H^{s-1}(\mathbb{R}^N)$  of  $P_\alpha(g, \partial_t)u$  for all  $u \in E_{T,M}$  as in the following.

**Lemma 3.2.** *Let  $P_\alpha$  be the operator defined by (3.1) and*

$$M = C \left( \|u_0\|_{H^s(\mathbb{R}^N)} + \|u_1\|_{H^{s-1}(\mathbb{R}^N)} \right).$$

*Then there exist two positive constants  $C_1$  and  $C_2$  depending only on  $T$  and  $M$  such that for all  $u \in E_{T,M}$  one has*

$$\int_0^t \|P_\alpha(g, \partial_t)u(\tau, \cdot)\|_{H^{s-1}(\mathbb{R}^N)} d\tau \leq C_1 M^m + C_2 T^{\alpha+1} M^p.$$

*Proof.* Since

$$\|u\|_{H^{s-1}(\mathbb{R}^N)} = \sum_{|\beta| \leq s-1} \|D_x^\beta u\|_{L^2(\mathbb{R}^N)},$$

then, using *Lemma 3.1*, we get for all  $\beta \in \mathbb{N}^N$  such that  $|\beta| \leq s-1$  and  $x \in \mathbb{R}^N$

$$\begin{aligned} D_x^\beta \int_0^t P_\alpha(g, \partial_t) u(\tau, x) d\tau &= \frac{1}{m} D_x^\beta [g(x) |u(0, x)|^{m-1} u(0, x)] \\ &\quad - \frac{1}{m} D_x^\beta (g(x) |u(t, x)|^{m-1} u(t, x)) \\ &\quad + \frac{1}{\alpha} D_x^\beta \int_0^t (t-\tau)^\alpha |u(\tau, x)|^p d\tau. \end{aligned}$$

Applying Lebesgue's dominated convergence theorem (Theorem 1.1.4, pp. 3 in [3]) and Leibnitz's formula (2.1) we obtain

$$\begin{aligned} \int_0^t \|P_\alpha(g, \partial_t) u(\tau, \cdot)\|_{H^{s-1}(\mathbb{R}^N)} d\tau &\leq \|g |u(0, \cdot)|^{m-1} u(0, \cdot)\|_{H^{s-1}(\mathbb{R}^N)} \\ &\quad + \|g |u(t, \cdot)|^{m-1} u(t, \cdot)\|_{H^{s-1}(\mathbb{R}^N)} \\ &\quad + \int_0^t (t-\tau)^\alpha \| |u(\tau, \cdot)|^p \|_{H^{s-1}(\mathbb{R}^N)} d\tau. \end{aligned}$$

By Sobolev's embedding theorem (*Lemma 2.1*) and *Lemma 2.4* we find

$$\begin{aligned} \int_0^t \|P_\alpha(g, \partial_t) u(\tau, \cdot)\|_{H^{s-1}(\mathbb{R}^N)} d\tau &\leq C \|g |u(0, \cdot)|^{m-1} u(0, \cdot)\|_{H^{s-1}(\mathbb{R}^N)} \\ &\quad + C \|g |u(t, \cdot)|^{m-1} u(t, \cdot)\|_{H^{s-1}(\mathbb{R}^N)} \\ &\quad + C \int_0^t (t-\tau)^\alpha \| |u(\tau, \cdot)|^p \|_{H^{s-1}(\mathbb{R}^N)} d\tau \\ &\leq C \|u\|_{H^s(\mathbb{R}^N)}^m + C \sup_{0 \leq t \leq T} \|u(t, \cdot)\|_{H^s(\mathbb{R}^N)}^m \\ &\quad + \sup_{0 \leq t \leq T} \|u(t, \cdot)\|_{H^s(\mathbb{R}^N)}^p \sup_{0 \leq t \leq T} \int_0^t (t-\tau)^\alpha d\tau \\ &\leq C_1 M^m + C_2 T^{\alpha+1} M^p. \end{aligned}$$

Hence the proof is completed.  $\square$

**Remark 3.2.** Let  $T > 0$ . Suppose that  $u \in X_T$  is a solution of the following Cauchy problem

$$(P2) \quad \begin{cases} u_{tt}(t, x) - \Delta u(t, x) = P_\alpha(g, \partial_t) v(t, x), & t > 0, x \in \mathbb{R}^n \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n \\ u_t(0, x) = u_1(x), & x \in \mathbb{R}^n. \end{cases}$$

Then for all  $v \in E_T$  the mapping  $\Phi$  defined by  $\Phi(v) = u$  is well defined.

*Proof.* Note that since  $g$  is bounded then for any  $v \in E_T$  we have by Sobolev's embedding theorem that  $P_\alpha(g, \partial_t)v \in L^\infty([0, T]; H^s(\mathbb{R}^N))$  which implies the existence and uniqueness of such  $u$  in  $E_T$  by the theory of mixed Cauchy problems for linear wave equations.  $\square$

Now using *Theorem 3.1* and *Lemma 3.2* we obtain the following results.

**Proposition 3.1.** *If  $v \in E_{T,M}$  then  $\Phi(v) = u \in X_{T,M}$ .*

*Proof.* Assume that  $v \in E_{T,M}$  and by choosing

$$M = C_0 \left( \|u_0\|_{H^s(\mathbb{R}^N)} + \|u_t\|_{H^{s-1}(\mathbb{R}^N)} \right), \quad C_0 > 0,$$

it follows by using *Lemma 3.2* and the theory of linear wave equations

$$\begin{aligned} \sup_{0 \leq t \leq T} (\|u(t, \cdot)\|_{H^s} + \|u_t(t, \cdot)\|_{H^{s-1}}) &\leq C(1+T) (\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}}) \\ &\quad + C(1+T) \left( \int_0^t \|P_\alpha(g, \partial_t)v(\tau, \cdot)\|_{H^{s-1}} d\tau \right) \\ &\leq C(T, M)M, \end{aligned}$$

where

$$C(T, M) = C(1+T) \left( \frac{1}{C_0} + CM^{m-1} + CT^{\alpha+1}M^{p-1} \right).$$

Since we can find  $T_1 > 0$  such that  $\forall T \in [0, T_1]$ ,  $C(T, M) \leq 1$ , we deduce that

$$\sup_{0 \leq t \leq T} \left( \|u\|_{H^s(\mathbb{R}^N)} + \|u_t\|_{H^{s-1}(\mathbb{R}^N)} \right) \leq M.$$

This ends the proof.  $\square$

**Proposition 3.2.** *The mapping  $\Phi$  is a contraction from  $X_{T,M}$  into  $X_{T,M}$ .*

*Proof.* Since the function  $(x, y) \mapsto |x|^{m-1}y$  is not Lipschitz continuous with respect to  $(x, y) \in \mathbb{R}^2$  for  $1 < m < 2$ , we cannot apply the mean value theorem directly. To overcome this obstacle, we will use the linearity and we modify the technique as it has been done by Berbiche and Hakem [8], Katayama [13], MD. Abu Naim [14] and by Lions and Strauss [19]. Let  $v_1, v_2 \in E_{T,M}$  such that  $v_1(0, x) = v_2(0, x) = u_0(x)$ ,  $x \in \mathbb{R}^N$  and let  $w, \nu$  be solutions for the following problems, respectively

$$(P.1.1) \quad \begin{cases} w_{tt}(t, x) - \Delta w(t, x) = \int_0^t (t-s)^{\alpha-1} |v_1(s, x)|^p ds, & t > 0, x \in \mathbb{R}^n \\ w(0, x) = u_0(x) & x \in \mathbb{R}^n \\ w_t(0, x) = u_1(x) + \frac{g(x)}{m} |u_0|^{m-1} u_0 & x \in \mathbb{R}^n, \end{cases}$$



and

$$(P.1.2) \quad \begin{cases} \nu_{tt}(t, x) - \Delta \nu(t, x) = \int_0^t (t-s)^{\alpha-1} |v_2(s, x)|^p ds, & t > 0, x \in \mathbb{R}^n \\ \nu(0, x) = u_0(x) & x \in \mathbb{R}^n \\ \nu_t(0, x) = u_1(x) + \frac{g(x)}{m} |u_0|^{m-1} u_0 & x \in \mathbb{R}^n. \end{cases}$$

Denote also by  $\tilde{w}$  and  $\tilde{\nu}$  the solutions of the following problems, respectively

$$(P.2.1) \quad \begin{cases} \tilde{w}_{tt}(t, x) - \Delta \tilde{w}(t, x) = -\frac{g(x)}{m} |v_1(t, x)|^{m-1} v_1(t, x), & t > 0, x \in \mathbb{R}^n \\ \tilde{w}(0, x) = 0 & x \in \mathbb{R}^n \\ \tilde{w}_t(0, x) = 0 & x \in \mathbb{R}^n, \end{cases}$$

and

$$(P.2.2) \quad \begin{cases} \tilde{\nu}_{tt}(t, x) - \Delta \tilde{\nu}(t, x) = -\frac{g(x)}{m} |v_2(t, x)|^{m-1} v_2(t, x), & t > 0, x \in \mathbb{R}^n \\ \tilde{\nu}(0, x) = 0 & x \in \mathbb{R}^n \\ \tilde{\nu}_t(0, x) = 0 & x \in \mathbb{R}^n. \end{cases}$$

**Remark 3.3.** The fact that  $g$  is bounded with all its derivatives and  $v_i \in E_{T,M}$  implies that for  $i = 1, 2$  we have

$$\begin{aligned} & \int_0^t (t-s)^{\alpha-1} |v_i(s, x)|^p ds, \quad \frac{g(x)}{m} |v_i(t, x)|^{m-1} v_i(t, x) \quad \text{and} \\ & \frac{g(x)}{m} |v_i(t, x)|^{m-1} \partial_t v_i(t, x) \in L^\infty \left( [0, T]; H^{s-1}(\mathbb{R}^N) \right), \end{aligned}$$

consequently, by Sobolev's embedding theorem, we deduce that

$$w, \nu \in E_{T,M}$$

and

$$\tilde{w}, \tilde{\nu} \in C \left( [0, T]; H^{s+1}(\mathbb{R}^N) \right) \cap C^1 \left( [0, T]; H^s(\mathbb{R}^N) \right) \cap C^2 \left( [0, T]; H^{s-1}(\mathbb{R}^N) \right).$$

Then we have the following results:

**Proposition 3.3.** Denoting  $\bar{w} = w + \tilde{w}_t$ , then  $\bar{w}$  is solution for the problem

$$(\bar{P}_1) \quad \begin{cases} \bar{w}_{tt}(t, x) - \Delta \bar{w}(t, x) = P_\alpha(g, \partial_t) v_1(t, x), & t > 0, x \in \mathbb{R}^n \\ \bar{w}(0, x) = u_0(x) & x \in \mathbb{R}^n \\ \bar{w}_t(0, x) = u_1(x) & x \in \mathbb{R}^n. \end{cases}$$

*Proof.* One can remark that

$$\text{for all } v_1 \in E_T \text{ and } t \in [0, T], \quad \partial_t \left( |v_1|^{m-1} v_1 \right) = m |v_1|^{m-1} \partial_t v_1,$$

hence a simple computation shows that

$$\begin{aligned}
 \partial_t^2 \bar{w} - \Delta \bar{w} &= \partial_t^2 (w + \tilde{w}_t) - \Delta (w + \tilde{w}_t) \\
 &= w_{tt}(t, x) - \Delta w(t, x) - \partial_t (\tilde{w}_{tt}(t, x) - \Delta \tilde{w}(t, x)) \\
 &= \int_0^t (t-s)^{\alpha-1} |v_1(t, x)|^p ds - g(x) |v_1(t, x)|^{m-1} v_1(t, x) \\
 &= P_\alpha(g, \partial_t) v_1(t, x).
 \end{aligned}$$

It is easy to see that the initial conditions are satisfied too as follows. We have firstly

$$\bar{w}(0, x) = w(0, x) + \tilde{w}_t(0, x) = u_0(x) \quad \text{for all } x \in \mathbb{R}^N.$$

Secondly, we have

$$\tilde{w}_{tt}(t, x) - \Delta \tilde{w}(t, x) = -\frac{g(x)}{m} |v_1(t, x)|^{m-1} v_1(t, x), \quad t \geq 0, x \in \mathbb{R}^N,$$

then for  $t = 0$ , we get

$$(3.6) \quad \tilde{w}_{tt}(0, x) - \Delta \tilde{w}(0, x) = -\frac{g(x)}{m} |v_1(0, x)|^{m-1} v_1(0, x),$$

and since  $\Delta \tilde{w}(0, x) = 0$  because  $\tilde{w}(0, x) = 0$  and  $v_1(0, x) = u_0(x)$ , we obtain from (3.6)

$$(3.7) \quad \tilde{w}_{tt}(0, x) = -\frac{g(x)}{m} |u_0(x)|^{m-1} u_0(x).$$

Now, using the formula (3.7) we find

$$\begin{aligned}
 \bar{w}_t(0, x) &= w_t(0, x) + \tilde{w}_{tt}(0, x) \\
 &= u_1(x) + \frac{g(x)}{m} |u_0(x)|^{m-1} u_0(x) - \frac{g(x)}{m} |u_0(x)|^{m-1} u_0(x) \\
 &= u_1(x).
 \end{aligned}$$

This completes the proof of *Proposition 3.3*.  $\square$

**Lemma 3.3.** Denoting  $\bar{\nu} = \nu + \tilde{\nu}_t$ , then  $\bar{\nu}$  is solution for the following problem

$$(\bar{P}_2) \quad \begin{cases} \bar{\nu}_{tt}(t, x) - \Delta \bar{\nu}(t, x) = P_\alpha(g, \partial_t) v_2(t, x), & t > 0, x \in \mathbb{R}^n \\ \bar{\nu}(0, x) = u_0(x), & x \in \mathbb{R}^n \\ \bar{\nu}_t(0, x) = u_1(x), & x \in \mathbb{R}^n. \end{cases}$$

*Proof.* The proof is similar to the proof of *Proposition 3.3*.  $\square$

Now by *Proposition 3.3*, *Lemma 3.3* and the definition of  $\Phi$ , we have the following corollary:

**Corollary 3.1.** *One has  $\Phi(v_1) = w + \tilde{w}_t$  and  $\Phi(v_2) = \nu + \tilde{\nu}_t$ .*

*Proof.* An immediate consequence of the definition of  $\Phi$ , Proposition 3.3, Lemma 3.3 and the uniqueness of the solution to linear wave equations.  $\square$

In order to prove that  $\Phi$  is a contraction mapping into  $E_{T,M}$ , we also need the following.

**Proposition 3.4.** *Denoting  $w^* = w - \nu$ , then there exists a constant  $C > 0$  such that*

$$\|w^*(t, \cdot)\|_{H^s} + \|w_t^*(t, \cdot)\|_{H^{s-1}} \leq C(1+T)T^{\alpha+1}M^{p-1} \sup_{0 \leq t \leq T} \|v_1(t, \cdot) - v_2(t, \cdot)\|_{H^s}.$$

*Proof.* First of all, we show that  $w^*$  is a solution for the following homogenous Cauchy problem:

$$\begin{cases} \partial_t^2 w^* - \Delta w^* = \int_0^t (t-s)^{-\gamma} (|v_1(s, \cdot)|^p - |v_2(s, \cdot)|^p) ds \\ w^*(0, x) = w_t^*(0, x) = 0. \end{cases}$$

To do this, it is enough to note that firstly

$$\begin{aligned} \partial_t^2 w^* - \Delta w^* &= w_{tt} - \Delta w - (\nu_{tt} - \Delta \nu) \\ &= \int_0^t (t-s)^{-\gamma} |v_1(s, x)|^p ds - \int_0^t (t-s)^{-\gamma} |v_2(s, x)|^p ds \\ &= \int_0^t (t-s)^{-\gamma} (|v_1(s, \cdot)|^p - |v_2(s, \cdot)|^p) ds, \end{aligned}$$

and secondly

$$w^*(0, x) = w(0, x) - \nu(0, x) = u_0(0, x) - u_0(0, x) = 0.$$

In the same way, we prove that  $w_t^*(0, x) = 0$ , hence, by the theory of linear wave equations, we get

$$\|w^*(t, \cdot)\|_{H^s} + \|w_t^*(t, \cdot)\|_{H^{s-1}} \leq C(1+T) \int_0^t \int_0^\tau (\tau-s)^{-\gamma} \| |v_1(s, \cdot)|^p - |v_2(s, \cdot)|^p \|_{H^{s-1}} ds d\tau.$$

Applying Fubini's theorem, we arrive at

$$\begin{aligned} &\|w^*(t, \cdot)\|_{H^s} + \|w_t^*(t, \cdot)\|_{H^{s-1}} \\ &\leq C(1+T) \int_0^t \left( \int_s^t (\tau-s)^{-\gamma} d\tau \right) \| |v_1(s, \cdot)|^p - |v_2(s, \cdot)|^p \|_{H^{s-1}} ds \\ &\leq C(1-\gamma)^{-1}(1+T) \int_0^t (t-s)^{1-\gamma} \| |v_1(s, \cdot)|^p - |v_2(s, \cdot)|^p \|_{H^{s-1}} ds. \end{aligned}$$

By the mean value theorem and Sobolev's embedding theorem, we obtain

$$\begin{aligned}
& \|w^*(t, \cdot)\|_{H^s} + \|w_t^*(t, \cdot)\|_{H^{s-1}} \\
& \leq C(1 - \gamma)^{-1} (1 + T) \sup_{0 \leq t \leq T} \left( \|v_1(t, \cdot)\|_{H^s}^{p-1} + \|v_2(t, \cdot)\|_{H^s}^{p-1} \right) \\
& \quad \times \sup_{0 \leq t \leq T} \|v_1(t, \cdot) - v_2(t, \cdot)\|_{H^s} \sup_{0 \leq t \leq T} \int_0^t (t-s)^{1-\gamma} ds \\
& \leq C(1 + T) T^{-\gamma} M^{p-1} \sup_{0 \leq t \leq T} \|v_1(t, \cdot) - v_2(t, \cdot)\|_{H^s}.
\end{aligned}$$

This ends the proof.  $\square$

**Proposition 3.5.** *Denoting  $\tilde{w}^* = \tilde{w} - \tilde{v}$ , then there exists a constant  $C > 0$  such that, for all  $t \in [0, T]$  we have*

$$\|\tilde{w}^*(t, \cdot)\|_{H^s} + \|\tilde{w}_t^*(t, \cdot)\|_{H^{s-1}} \leq C(1 + T) T M^{m-1} \sup_{0 \leq t \leq T} \|v_1(t, \cdot) - v_2(t, \cdot)\|_{H^s}.$$

*Proof.* It is easy to show that  $\tilde{w}^*$  is a solution to the following Cauchy problem

$$\begin{cases} \partial_t^2 \tilde{w}^* - \Delta \tilde{w}^* = -\frac{g}{m} (|v_1(s, \cdot)|^{m-1} v_1 - |v_2(s, \cdot)|^{m-1} v_2) \\ \tilde{w}^*(0, x) = \tilde{w}_t^*(0, x) = 0, \end{cases}$$

hence, by the theory of linear wave equations, we find

$$\begin{aligned}
& \|\tilde{w}^*(t, \cdot)\|_{H^s} + \|\tilde{w}_t^*(t, \cdot)\|_{H^{s-1}} \\
& \leq C(1 + T) \int_0^t \left\| \frac{g}{m} |v_1(s, \cdot)|^{m-1} v_1 - \frac{g}{m} |v_2(s, \cdot)|^{m-1} v_2 \right\|_{H^{s-1}} ds,
\end{aligned}$$

by using *Lemma.2.4*, we arrive at

$$\begin{aligned}
& \|\tilde{w}^*(t, \cdot)\|_{H^s} + \|\tilde{w}_t^*(t, \cdot)\|_{H^{s-1}} \\
& \leq C(1 + T) \int_0^t \left\| |v_1(s, \cdot)|^{m-1} v_1 - |v_2(s, \cdot)|^{m-1} v_2 \right\|_{H^{s-1}} ds.
\end{aligned}$$

Taking into account the mean value theorem, we get

$$\begin{aligned}
& \|\tilde{w}^*(t, \cdot)\|_{H^s} + \|\tilde{w}_t^*(t, \cdot)\|_{H^{s-1}} \leq \\
& C(1 + T) \int_0^t \left( \|v_1(s, \cdot)\|_{H^s}^{m-1} + \|v_2(s, \cdot)\|_{H^s}^{m-1} \right) \|v_1(s, \cdot) - v_2(s, \cdot)\|_{H^s} ds \\
(3.8) \quad & \leq C(1 + T) \sup_{0 \leq t \leq T} \left( \|v_1(t, \cdot)\|_{H^s}^{m-1} + \|v_2(t, \cdot)\|_{H^s}^{m-1} \right) \\
& \quad \times \sup_{0 \leq t \leq T} \|v_1(t, \cdot) - v_2(t, \cdot)\|_{H^s} \sup_{0 \leq t \leq T} \int_0^t ds \\
& \leq C(1 + T) T M^{m-1} \sup_{0 \leq t \leq T} \|v_1(t, \cdot) - v_2(t, \cdot)\|_{H^s}.
\end{aligned}$$

This completes the proof.  $\square$

Now we are able to estimate the  $H^s$ -norm of the quantity  $\Phi(v_1) - \Phi(v_2)$  and show that  $\Phi$  is a contraction mapping as follows. For all  $v_1, v_2 \in E_{T,M}$  and  $t \in [0, T]$ , we have

$$\begin{aligned} \|\Phi(v_1) - \Phi(v_2)\|_{H^s} &= \|w(t, \cdot) + \tilde{w}_t(t, \cdot) - \nu(t, \cdot) - \tilde{\nu}_t(t, \cdot)\|_{H^s} \\ &\leq \|w - \nu\|_{H^s} + \|\tilde{w}_t - \tilde{\nu}_t\|_{H^s} \\ &\leq \|w^*(t, \cdot)\|_{H^s} + \|w_t^*(t, \cdot)\|_{H^{s-1}} + \|\tilde{w}^*(t, \cdot)\|_{H^s} + \|\tilde{w}_t^*(t, \cdot)\|_{H^{s-1}} \\ &\leq C(1+T)T^{\alpha+1}M^{p-1} \sup_{0 \leq t \leq T} \|v_1(t, \cdot) - v_2(t, \cdot)\|_{H^s} \\ &\quad + C(1+T)TM^{m-1} \sup_{0 \leq t \leq T} \|v_1(t, \cdot) - v_2(t, \cdot)\|_{H^s} \\ &\leq C(1+T)[T^{\alpha+1}M^{p-1} + TM^{m-1}] \sup_{0 \leq t \leq T} \|v_1(t, \cdot) - v_2(t, \cdot)\|_{H^s}, \end{aligned}$$

hence

$$(3.9) \quad \sup_{0 \leq t \leq T} \|\Phi(v_1) - \Phi(v_2)\|_{H^s} \leq C(1+T)[T^{\alpha+1}M^{p-1} + TM^{m-1}] \sup_{0 \leq t \leq T} \|v_1(t, \cdot) - v_2(t, \cdot)\|_{H^s}.$$

Since it is possible to find  $T_1 > 0$  satisfying

$$C(1+T)[T^{\alpha+1}M^{p-1} + TM^{m-1}] < 1, \quad \forall T \in [0, T_1],$$

we deduce from (3.9) that

$$(3.10) \quad \sup_{0 \leq t \leq T} \|\Phi(v_1) - \Phi(v_2)\|_{H^s} \leq k \sup_{0 \leq t \leq T} \|v_1(t, \cdot) - v_2(t, \cdot)\|_{H^s}, \quad k \in ]0, 1[.$$

Now, using *Remark 3.1* and *Proposition 3.1*, we easily show that

$$\Phi(X_{T,M}) \subset X_{T,M}.$$

Finally, define a sequence  $(u^{(n)})_n$  as follows

$$(3.11) \quad \begin{cases} u^{(0)}(t, x) = u(0, x) = u_0(x), \\ u^{(n)}(t, x) = \Phi(u^{(n-1)})(t, x). \end{cases}$$

By (3.10), for all  $T > 0$  there exists some  $\bar{u} \in C([0, T], H^s)$  such that  $u^{(n)} \rightarrow \bar{u}$  in  $C([0, T], H^s)$  as  $n \rightarrow \infty$ . The aim now is to show that this  $\bar{u}$  belongs to  $X_T$  and is a solution to *problem (1.3)*. Since  $u^{(n)} \in X_{T,M}$ , then  $(u^{(n)})_n$  and  $(u_t^{(n)})_n$  has a weak convergent subsequence  $(u^{(n_k)})_k$  (resp.  $(u_t^{(n_k)})_k$ ) in  $L^\infty([0, T]; H^s)$  (resp. in  $L^\infty([0, T]; H^{s-1})$ ). Since  $(u^{(n)})_n$  converges to  $\bar{u}$  in  $C([0, T], H^s)$ , the above subsequence converges weakly to  $\bar{u}$  in  $L^\infty([0, T]; H^s)$  (resp. in  $L^\infty([0, T]; H^{s-1})$ ),

as a consequence, we see that  $\bar{u} \in L^\infty([0, T]; H^s)$  and  $\bar{u}_t \in L^\infty([0, T]; H^{s-1})$ , this means that  $\bar{u} \in E_{T,M}$  and then  $\Phi(\bar{u}) \in X_{T,M}$ . Applying (3.10) we get for  $k \in ]0, 1[$

$$(3.12) \quad \sup_{0 \leq t \leq T} \left\| \Phi(u^{(n)}) - \Phi(\bar{u}) \right\|_{H^s} \leq k \sup_{0 \leq t \leq T} \left\| u^{(n)}(t, \cdot) - \bar{u}(t, \cdot) \right\|_{H^s}.$$

Since the right-hand side of (3.12) goes to 0 as  $n \rightarrow +\infty$ , then  $\Phi(u^{(n)})$  converges to  $\Phi(\bar{u})$  in  $C([0, T], H^s)$ . Passing to the limit in  $u^{(n)} = \Phi(u^{(n-1)})$  as  $n \rightarrow +\infty$  and using the fact that  $u_n^{(n)} \rightarrow \bar{u}$  in  $C([0, T], H^s)$ , we obtain

$$\Phi(\bar{u}) = \bar{u} \in X_{T,M}.$$

This  $\bar{u}$  is apparently the desired solution. The uniqueness of such a solution in  $X_{T,M}$  follows immediately from the formula (3.10). This achieves the proof of Theorem 3.1.  $\square$

#### 4. Blow-up results of solutions for problem (1.3)

In this section, we will investigate the blow-up results of *problem* (1.3).

##### 4.1. Notations and definitions

**Definition 4.1.** Let  $u_0 \in L^1_{loc}(\mathbb{R}^N) \cap L^m_{loc}(\mathbb{R}^N)$  and  $u_1 \in L^1_{loc}(\mathbb{R}^N)$  be given. We say that  $u$  is a weak solution to the problem (1.3) if  $u \in L^p((0, T), L^p_{loc}(\mathbb{R}^N)) \cap L^m((0, T), L^m_{loc}(\mathbb{R}^N))$  and satisfies the following formula

$$(4.1) \quad \begin{aligned} & \Gamma(\alpha) \int_0^T \int_{\mathbb{R}^n} I_{0|t}^\alpha(|u|^p) \varphi(t, x) dt dx + \int_{\mathbb{R}^n} u_1(x) \varphi(0, x) dx \\ & - \int_{\mathbb{R}^n} u_0(x) \varphi_t(0, x) dx + \frac{1}{m} \int_{\mathbb{R}^n} g(x) |u_0|^{m-1}(x) u_0(x) \varphi(0, x) dx \\ & = \int_0^T \int_{\mathbb{R}^n} u(t, x) \varphi_{tt}(t, x) dt dx - \int_0^T \int_{\mathbb{R}^n} u(t, x) \Delta \varphi(t, x) dt dx, \\ & - \frac{1}{m} \int_0^T \int_{\mathbb{R}^n} g(x) (|u|^{m-1} u)(t, x) \varphi_t(t, x) dt dx \end{aligned}$$

for all non-negative test functions  $\varphi \in C^2([0, T] \times \mathbb{R}^N)$  such that  $\varphi_t(T, \cdot) = \varphi(T, \cdot) = 0$  and  $\alpha = 1 - \gamma$ .

The main result of this section is the following theorem.

**Theorem 4.1.** Let  $0 < \gamma < 1$  and  $p, m \in \mathbb{R}$  such that  $p > m > 1$ . Assume that

$$(4.2) \quad \int_{\mathbb{R}^n} u_0(x) dx > 0, \quad \int_{\mathbb{R}^n} g(x) |u_0(x)|^m dx > 0, \quad \int_{\mathbb{R}^n} u_1(x) dx > 0.$$

Then the solution of the Cauchy problem (1.3) does not exist globally in time if one of the following conditions is fulfilled:

1.

$$N \leq \min \left( \frac{2((1-\gamma)p+m)}{(1-\gamma)(m-1)+(p-1)}, -\frac{2p(2-\gamma)+2}{\left(\frac{(2-\gamma)(p-1)}{m-p}+(1-\gamma)\right)(p-1)} \right)$$

2.  $p < \frac{1}{\gamma}$  or  $p = \frac{1}{\gamma}$  and moreover  $\frac{N-2}{N} < \gamma < 1$ .

*Proof.* The theorem (4.1) will be proved by absurd, so we suppose that  $u$  is a global non-trivial weak solution to the problem (1.3). To prove Theorem 4.1 we also need some results that we will give in the following section.  $\square$

## 4.2. Preliminary results

Since the principle of the method is the right choice of the test function, we choose it as follows

$$(4.3) \quad \varphi(t, x) = D_{t|T}^\alpha \psi(t, x) = \varphi_1^r(x) D_{t|T}^\alpha \varphi_2(t), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N,$$

where  $D_{t|T}^\alpha$  is the right fractional derivative operator in the sense of Riemann-Liouville defined by

$$(4.4) \quad D_{t|T}^\alpha v(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_t^T \frac{v(s)}{(s-t)^\alpha} ds,$$

and the functions  $\varphi_1$  and  $\varphi_2$  are given by

$$(4.5) \quad \varphi_1(x) = \phi\left(\frac{x^2}{T^\theta}\right) \quad \text{and} \quad \varphi_2(t) = \sup \left\{ 0, \left(1 - \frac{t}{T}\right)^\beta \right\},$$

with  $\beta \in \mathbb{R}_+^*$ ,  $\theta$  is a nonnegative parameter which will be specified later and  $\phi$  is a cut-off non-increasing function satisfying

$$(4.6) \quad \phi(s) = \begin{cases} 1 & \text{if } 0 \leq s \leq 1 \\ 0 & \text{if } s \geq 2 \end{cases}, \quad 0 \leq \phi \leq 1 \text{ everywhere and } \phi'(s) \leq \frac{C}{s}.$$

We will also use the fractional version of the integration by parts (See [18])

$$(4.7) \quad \int_0^t f(t) D_{t|T}^\alpha g(t) dt = \int_0^t \left( D_{0|t}^\alpha f(t) \right) g(t) dt,$$

for all  $f, g \in \mathcal{C}([0, T])$  such that  $D_{0|t}^\alpha(f(t))$  and  $D_{t|T}^\alpha g(t)$  exist and are continuous, and the following identity

$$(4.8) \quad \left( D_{0|t}^\alpha \circ I_{0|t}^\alpha \right) (u) = u \text{ for all } u \in L^q([0, T])$$

and also the bellow identity (see [18])

$$(4.9) \quad (-1)^n \partial_t^n D_{t|T}^\alpha u(t) = D_{t|T}^{\alpha+n} u(t), \quad n \in \mathbb{N}, \quad \alpha \in ]0, 1[$$

which occurs for all  $u \in C^n[0, T]; T > 0$ , where  $\partial_t^n$  is the  $n$ -times ordinary derivative with respect to  $t$  that will be useful in this paper.

A simple and immediate computation leads to

**Proposition 4.1.** *Given  $\beta > 0$ . Let  $\varphi_2$  be the function defined by*

$$\varphi_2(t) = \left(1 - \frac{t}{T}\right)_+^\beta,$$

then, for all  $\alpha \in ]0, 1[$ , we have

$$\begin{aligned} D_{t|T}^\alpha \varphi_2(t) &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} T^{-\beta} (T-t)_+^{\beta-\alpha} \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} T^{-\alpha} \left(1 - \frac{t}{T}\right)_+^{\beta-\alpha} \end{aligned}$$

$$\begin{aligned} D_{t|T}^{\alpha+1} \varphi_2(t) &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha)} T^{-\beta} (T-t)_+^{\beta-\alpha-1} \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha)} T^{-\alpha-1} \left(1 - \frac{t}{T}\right)_+^{\beta-\alpha-1} \end{aligned}$$

$$\begin{aligned} D_{t|T}^{\alpha+2} \varphi_2(t) &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha-1)} T^{-\beta} (T-t)_+^{\beta-\alpha-2} \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha-1)} T^{-\alpha-2} \left(1 - \frac{t}{T}\right)_+^{\beta-\alpha-2} \end{aligned}$$

*Proof.* The proof of *Proposition 4.1* is a simple and immediate verification. We get from the formula (4.4)

$$D_{t|T}^\alpha \varphi_2(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_t^T \frac{\varphi_2(s)}{(s-t)^\alpha} ds.$$

Using Euler's change of the variable

$$s \mapsto y = \frac{s-t}{T-t}$$



1. We have firstly

$$\begin{aligned}
 D_{t|T}^\alpha \varphi_2(t) &= \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_t^T \frac{(1-\frac{s}{T})^\beta}{(s-t)^\alpha} ds \\
 &= \frac{T^{-\beta}}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \left( (T-t)^{\beta-\alpha+1} \int_0^1 y^{-\alpha} (1-y)^\beta dy \right) \\
 &= \frac{(\beta-\alpha+1) \mathcal{B}(1-\alpha, \beta+1)}{\Gamma(1-\alpha)} T^{-\beta} (T-t)^{\beta-\alpha} \\
 &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} T^{-\alpha} (T-t)^{\beta-\alpha},
 \end{aligned}$$

where  $\mathcal{B}$  is the famous *Béta function* defined by

$$\mathcal{B}(u, v) = \int_0^1 t^{u-1} (1-t)^{v-1} dt.$$

2. We directly apply the formula (4.9) to show that

$$\forall t \in [0, T] : D_{t|T}^{\alpha+1} \varphi_2(t) = -\partial_t D_{t|T}^\alpha \varphi_2(t) \text{ et } D_{t|T}^{\alpha+2} \varphi_2(t) = \partial_t^2 D_{t|T}^\alpha \varphi_2(t).$$

Hence the proof is completed.  $\square$

### 4.3. Treatment of the weak formulation (4.1)

#### 4.3.1. Treatment of the left-hand side

Using the parts integration formula (4.7) and the identity (4.8) we get

$$\begin{aligned}
 (4.10) \quad & \int_0^T \int_{\mathbb{R}^n} I_{0|t}^\alpha (|u|^p) \varphi(t, x) dt dx = \int_0^T \int_{\mathbb{R}^n} I_{0|t}^\alpha (|u|^p) D_{0|T}^\alpha \psi(t, x) dt dx \\
 &= \int_0^T \int_{\mathbb{R}^n} D_{0|T}^\alpha I_{0|T}^\alpha (|u|^p) \psi(t, x) dt dx = \int_0^T \int_{\mathbb{R}^n} |u|^p \psi(t, x) dt dx.
 \end{aligned}$$

For the  $2^{nd}$  term of the left-hand side of the equality (4.1), we use *Proposition 4.1*. We easily obtain

$$\begin{aligned}
 (4.11) \quad & \int_{\mathbb{R}^n} u_1(x) \varphi(0, x) dx = \int_{\mathbb{R}^n} u_1(x) \varphi_1^r(x) D_{t|T}^\alpha \varphi_2(t) \Big|_{t=0} dx \\
 &= C_1 T^{-\alpha} \int_{\mathbb{R}^n} u_1(x) \varphi_1^r(x) dx,
 \end{aligned}$$

since

$$D_{t|T}^\alpha \varphi_2(t) \Big|_{t=0} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} T^{-\alpha} = C_1 T^{-\alpha}.$$

For the third term, noting that

$$\varphi_t(t, x) = \frac{\partial \varphi}{\partial t}(t, x) = \varphi_1^r(x) D_{t|T}^{\alpha+1} \varphi_2(t),$$

using therefore *Proposition 4.1* we get the following estimate

$$(4.12) \quad \int_{\mathbb{R}^n} u_0(x) \varphi_t(0, x) dx = C_2 T^{-\alpha-1} \int_{\mathbb{R}^n} u_0(x) \varphi_1^r(x) dx,$$

since

$$D_{0|T}^{\alpha+1} \varphi_2(t) \Big|_{t=0} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha)} T^{-\alpha-1} = C_2 T^{-\alpha-1}.$$

Hence by *Proposition 4.1*, the following estimate will be obtained for the 4<sup>th</sup> term of the left-hand side of the weak formulation(4.1).

$$(4.13) \quad \begin{aligned} & \int_{\mathbb{R}^n} g(x) |u_0|^{m-1}(x) u_0(x) \varphi(0, x) dx \\ &= C_1 T^{-\alpha} \int_{\mathbb{R}^n} g(x) |u_0|^{m-1} u_0(x) \varphi_1^r(x) dx. \end{aligned}$$

#### 4.3.2. Treatment of the right-hand side

Making use of (4.9), it can be seen that

$$\varphi_{tt}(t, x) = \varphi_1^r(x) \partial_t^2 D_{t|T}^{\alpha} \varphi_2(t) = \varphi_1^r(x) D_{t|T}^{\alpha+2} \varphi_2(t),$$

one can deduce that

$$(4.14) \quad \int_0^T \int_{\mathbb{R}^n} u(t, x) \varphi_{tt}(t, x) dt dx = \int_0^T \int_{\mathbb{R}^n} u(t, x) \varphi_1^r(x) D_{t|T}^{\alpha+2} \varphi_2(t) dt dx.$$

Similarly, as

$$\varphi_t(t, x) = \varphi_1^r(x) \partial_t D_{t|T}^{\alpha} \varphi_2(t) = -\varphi_1^r(x) D_{t|T}^{\alpha+1} \varphi_2(t),$$

we get

$$(4.15) \quad \begin{aligned} & \int_0^T \int_{\mathbb{R}^n} g(x) |u|^{m-1} u(t, x) \varphi_t(t, x) dt dx = \\ & - \int_0^T \int_{\mathbb{R}^n} g(x) |u|^{m-1} u(x) \varphi_1^r(x) D_{t|T}^{\alpha+1} \varphi_2(t) dt dx. \end{aligned}$$

Finally, for the third term of the right-hand side of the formulation (4.1), we use the following identity

$$\Delta(\varphi_1^r) = r \varphi_1^{r-1} \Delta \varphi_1 + r(r-1) \varphi_1^{r-2} |\nabla \varphi_1|^2,$$

to obtain

$$(4.16) \quad \begin{aligned} & \int_0^T \int_{\mathbb{R}^n} u(t, x) \Delta \varphi(t, x) dt dx = \\ & \int_0^T \int_{\mathbb{R}^n} u(t, x) (r \varphi_1^{r-1} \Delta \varphi + r(r-1) \varphi_1^{r-2} |\nabla \varphi_1|^2) D_{t|T}^\alpha \varphi_2(t) dt dx. \end{aligned}$$

Inserting the formulas (4.10), (4.11), (4.12), (4.13), (4.14), (4.15) et (4.16) in the formula (4.1) we deduce

$$(4.17) \quad \begin{aligned} & \Gamma(\alpha) \int_0^T \int_{\mathbb{R}^n} |u|^p \psi(t, x) dt dx + C_1 T^{-\alpha} \int_{\mathbb{R}^n} u_1(x) \varphi_1^r(x) dx \\ & + C_2 T^{-\alpha-1} \int_{\mathbb{R}^n} u_0(x) \varphi_1^r(x) dx + \frac{C_1}{m} T^{-\alpha} \int_{\mathbb{R}^n} g(x) |u_0|^{m-1}(x) \varphi_1^r(x) dx \\ & = \int_0^T \int_{\mathbb{R}^n} u(t, x) \varphi_1^r(x) D_{t|T}^{\alpha+2} \varphi_2(t) dt dx \\ & - \int_0^T \int_{\mathbb{R}^n} u(t, x) \left[ r \varphi_1^{r-1} \Delta \varphi_1 + r(r-1) \varphi_1^{r-2} |\nabla \varphi_1|^2 \right] (x) \cdot D_{t|T}^\alpha \varphi_2(t) dt dx \\ & + \frac{1}{m} \int_0^T \int_{\mathbb{R}^n} g(x) |u|^m(x) \varphi_1^r(x) D_{t|T}^{\alpha+1} \varphi_2(t) dt dx. \end{aligned}$$

The fact that  $\varphi_1^r \leq 1$  and

$$\left| r \varphi_1^{r-1} \Delta \varphi_1 + r(r-1) \varphi_1^{r-2} |\nabla \varphi_1|^2 \right| \leq \varphi_1^{r-2} \left( |\Delta \varphi_1| + |\nabla \varphi_1|^2 \right),$$

allow us to get from the formula (4.17) the following inequality

$$(4.18) \quad \begin{aligned} & \int_0^T \int_{\mathbb{R}^n} |u|^p \psi(t, x) dt dx + C T^{-\alpha} \int_{\mathbb{R}^n} u_1(x) \varphi_1^r(x) dx \\ & + C T^{-\alpha-1} \int_{\mathbb{R}^n} u_0(x) \varphi_1^r(x) dx + C T^{-\alpha} \int_{\mathbb{R}^n} g(x) |u_0|^m(x) \varphi_1^r(x) dx \\ & \leq C \int_0^T \int_{\mathbb{R}^n} |u(t, x)| \varphi_1^r(x) \left| D_{t|T}^{\alpha+2} \varphi_2(t) \right| dt dx \\ & + C \int_0^T \int_{\mathbb{R}^n} |u(t, x)| \varphi_1^{r-2} \left[ |\Delta \varphi_1| + |\nabla \varphi_1|^2 \right] \left| D_{t|T}^\alpha \varphi_2(t) \right| dt dx \\ & + C \int_0^T \int_{\mathbb{R}^n} |u|^m g(x) \varphi_1^r(x) \left| D_{t|T}^{\alpha+1} \varphi_2(t) \right| dt dx, \end{aligned}$$

for some constant  $C > 0$ . Applying  $\varepsilon$ -Young inequality

$$AB \leq \varepsilon A^p + C(\varepsilon) B^q, \quad pq = p + q, \quad p, q > 1,$$

to terms of the right-hand side of the inequality (4.18), we find

$$\begin{aligned}
 & \int_0^T \int_{\mathbb{R}^n} |u(t, x)| \varphi_1^r(x) \left| D_{t|T}^{\alpha+2} \varphi_2(t) \right| dt dx = \\
 (4.19) \quad & \int_0^T \int_{\mathbb{R}^n} u(t, x) \psi^{\frac{1}{p}} \psi^{-\frac{1}{p}} \varphi_1^r(x) \left| D_{t|T}^{\alpha+2} \varphi_2(t) \right| dt dx \\
 & \leq \varepsilon \int_0^T \int_{\mathbb{R}^n} |u|^p \psi dt dx + C(\varepsilon) \int_0^T \int_{\mathbb{R}^n} \varphi_1^r \varphi_2^{-\frac{1}{p-1}} \left| D_{t|T}^{\alpha+2} \varphi_2 \right|^{\frac{p}{p-1}} dt dx.
 \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 & \int_0^T \int_{\mathbb{R}^n} |u| \varphi_1^{r-2} \left( |\Delta \varphi_1| + |\nabla \varphi_1|^2 \right) \left| D_{t|T}^{\alpha} \varphi_2 \right| dt dx \leq \\
 (4.20) \quad & \varepsilon \int_0^T \int_{\mathbb{R}^n} |u|^p \psi dt dx + C(\varepsilon) \int_0^T \int_{\mathbb{R}^n} \mu(\varphi_1) \varphi_1^{r-2q} \varphi_2^{-\frac{1}{p-1}} \left| D_{t|T}^{\alpha} \varphi_2 \right|^q dt dx,
 \end{aligned}$$

with  $\mu(\varphi_1) = |\Delta \varphi_1|^q + |\nabla \varphi_1|^{2q}$ . For the third term of the right-hand side, we get

$$\begin{aligned}
 & \int_0^T \int_{\mathbb{R}^n} |g| |u|^m \varphi_1^r(x) \left| D_{t|T}^{\alpha+1} \varphi_2(t) \right| dt dx = \\
 (4.21) \quad & \int_0^T \int_{\mathbb{R}^n} |u|^m \psi^{\frac{m}{p}} \psi^{-\frac{m}{p}} |g| \varphi_1^r(x) \left| D_{t|T}^{\alpha+1} \varphi_2(t) \right| dt dx \\
 & \leq \varepsilon \int_0^T \int_{\mathbb{R}^n} |u|^p \psi dt dx \\
 & + C(\varepsilon) \int_0^T \int_{\mathbb{R}^n} \varphi_1^r |\varphi_2|^{-\frac{m}{p-m}} |g| \left| D_{t|T}^{\alpha+1} \varphi_2(t) \right|^{\frac{p}{p-m}} dt dx.
 \end{aligned}$$

Using (4.2) and the fact that

$$(4.22) \quad \int_{\mathbb{R}^n} u_i(x) \varphi_1^r(x) dx > 0, i = 0, 1 \text{ and } \int_{\mathbb{R}^n} g(x) |u_0|^m(x) \varphi_1^r(x) dx > 0,$$

we deduce from (4.18), (4.19), (4.20) and (4.21), for  $\varepsilon$  small enough

$$\begin{aligned}
 & \int_0^T \int_{\mathbb{R}^n} |u|^p \psi(t, x) dt dx \leq C \left( \int_0^T \int_{\mathbb{R}^n} \varphi_1^r \varphi_2^{-\frac{1}{p-1}} \left| D_{t|T}^{\alpha+2} \varphi_2 \right|^{\frac{p}{p-1}} dt dx \right. \\
 (4.23) \quad & + \int_0^T \int_{\mathbb{R}^n} \mu(\varphi_1) \varphi_1^{r-2q} \varphi_2^{-\frac{1}{p-1}} \left| D_{t|T}^{\alpha} \varphi_2 \right|^q dt dx \\
 & + \left. \int_0^T \int_{\mathbb{R}^n} \varphi_1^r |\varphi_2|^{\frac{m}{p-m}} |g| \left| D_{t|T}^{\alpha+1} \varphi_2(t) \right|^{\frac{p}{p-m}} dt dx \right) \\
 & \leq C(I_1 + I_2 + I_3),
 \end{aligned}$$

for some positive constant  $C$ . Now, to estimate the integrals  $I_1, I_2$  and  $I_3$ , at this stage we consider the scaled variables

$$(4.24) \quad x = T^{\frac{\theta}{2}}y \quad \text{and} \quad t = T\tau,$$

where  $\theta$  is the parameter which appears in the form of  $\varphi_1$  (page 643) and noting that  $I_1, I_2$  and  $I_3$  are null outside  $\Omega_T$  such that

$$\Omega_T := \left\{x \in \mathbb{R}^N, \quad |x|^2 \leq 2T^\theta\right\} = \text{supp}\varphi_1.$$

By Fubini's theorem (*Theorem 1.1.7 pp. 8 in [3]*), it is easy to see that

$$\begin{aligned} & \int_0^T \int_{\Omega_T} \varphi_1^r \varphi_2^{-\frac{1}{p-1}} \left| D_{t|T}^{\alpha+2} \varphi_2 \right|^{\frac{p}{p-1}} dt dx = \\ & \left( \int_{\Omega_T} \varphi_1^r dx \right) \left( \int_0^T \varphi_2^{-\frac{1}{p-1}} \left| D_{t|T}^{\alpha+2} \varphi_2 \right|^{\frac{p}{p-1}} dt \right) = J_{11} J_{12}. \end{aligned}$$

We have

$$(4.25) \quad J_{11} = \int_{\Omega_T} \varphi_1^r(x) dx = T^{\frac{N\theta}{2}} \int_0^2 \phi^r(y^2) dy = CT^{\frac{N\theta}{2}},$$

then using *Proposition 4.1*, we get

$$(4.26) \quad J_{12} = \int_0^T \varphi_2^{-\frac{1}{p-1}} \left| D_{t|T}^{\alpha+2} \varphi_2 \right|^{\frac{p}{p-1}} dt = CT^{1-(\alpha+2)\frac{p}{p-1}}.$$

Combining (4.25) and (4.26) into (4.25) we obtain

$$(4.27) \quad \int_0^T \int_{\Omega_T} \varphi_1^r \varphi_2^{-\frac{1}{p-1}} \left| D_{t|T}^{\alpha+2} \varphi_2 \right|^{\frac{p}{p-1}} dt dx = CT^{-(\alpha+2)\frac{p}{p-1} + \frac{N\theta}{2} + 1}.$$

In the same way we have

$$(4.28) \quad \begin{aligned} & \int_0^T \int_{\Omega_T} \mu(\varphi_1) \varphi_1^{r-2q} \varphi_2^{-\frac{1}{p-1}} \left| D_{t|T}^{\alpha} \varphi_2 \right|^q dt dx = \\ & \left( \int_{\Omega_T} \mu(\varphi_1) \varphi_1^{r-2q} dx \right) \left( \int_0^T \varphi_2^{-\frac{1}{p-1}} \left| D_{t|T}^{\alpha} \varphi_2 \right|^q dt \right) = J_{21} J_{22}. \end{aligned}$$

So, if we replace  $q$  by its value  $\frac{p}{p-1}$  we get

$$(4.29) \quad J_{21} = \int_{\Omega_T} \left( |\Delta \varphi_1|^{\frac{p}{p-1}} + |\nabla \varphi_1|^2 \right)^{\frac{p}{p-1}} \varphi_1^{r-2\frac{p}{p-1}} dx = CT^{-\theta\frac{p}{p-1} + \frac{N\theta}{2}},$$

and

$$(4.30) \quad J_{22} = \int_0^T \varphi_2^{-\frac{1}{p-1}} \left| D_{t|T}^{\alpha} \varphi_2 \right|^{\frac{p}{p-1}} dt = CT^{-\alpha\frac{p}{p-1} + 1}.$$

Plugging (4.29) and (4.30) into (4.28) we find

$$(4.31) \quad \int_0^T \int_{\Omega_T} \mu(\varphi_1) \varphi_1^{r-2q} \varphi_2^{-\frac{1}{p-1}} \left| D_{t|T}^\alpha \varphi_2 \right|^q dt dx = CT^{-(\alpha+\theta)\frac{p}{p-1} + \frac{N\theta}{2} + 1}.$$

For the third term we have

$$(4.32) \quad \int_0^T \int_{\Omega_T} \varphi_1^r |\varphi_2|^{-\frac{m}{p-m}} |g| \left| D_{t|T}^{\alpha+1} \varphi_2(t) \right|^{\frac{p}{p-m}} dt dx = \left( \int_{\Omega_T} |g| \varphi_1^r dx \right) \left( \int_0^T |\varphi_2|^{-\frac{m}{p-m}} \left| D_{t|T}^{\alpha+1} \varphi_2(t) \right|^{\frac{p}{p-m}} dt \right) = J_{31} J_{32}.$$

By the mean value theorem, we get for  $J_{31}$

$$(4.33) \quad \begin{aligned} J_{31} &\leq |g(\xi)| \int_{\Omega_T} \varphi_1^r dx \quad \text{for some } \xi \in \Omega_T \\ &= T^{\frac{N\theta}{2}} |g(\xi)| \int_0^2 \phi^r(y^2) dy = CT^{\frac{N\theta}{2}}. \end{aligned}$$

Since  $g$  is bounded, we have

$$(4.34) \quad J_{32} = \int_0^T |\varphi_2|^{-\frac{m}{p-m}} \left| D_{t|T}^{\alpha+1} \varphi_2(t) \right|^{\frac{p}{p-m}} dt = CT^{-(\alpha+1)\frac{p}{p-m} + 1}.$$

Hence inserting (4.33) and (4.34) in 4.32 we obtain

$$(4.35) \quad \int_0^T \int_{\Omega_T} \varphi_1^r |\varphi_2|^{-\frac{m}{p-m}} |g| \left| D_{t|T}^{\alpha+1} \varphi_2(t) \right|^{\frac{p}{p-m}} dt dx = CT^{-(\alpha+1)\frac{p}{p-m} + 1 + \frac{N\theta}{2}}.$$

Finally, we replace (4.27), (4.31) and (4.35) into (4.23) we get

$$(4.36) \quad \int_0^T \int_{\Omega_T} |u|^p \psi(t, x) dt dx \leq C \left( T^{-(\alpha+2)\frac{p}{p-1} + \frac{N\theta}{2} + 1} + T^{-(\alpha+\theta)\frac{p}{p-1} + \frac{N\theta}{2} + 1} + T^{-(\alpha+1)\frac{p}{p-m} + 1 + \frac{N\theta}{2}} \right).$$

Now, since  $\theta$  is arbitrary and it must only be nonnegative, we choose it as

$$(4.37) \quad \theta = \frac{(p-1)(\alpha+1)}{p-m} - \alpha > 0 \quad \text{since } p > m.$$

This choice of  $\theta$  allows us to have

$$(4.38) \quad -(\alpha+\theta)\frac{p}{p-1} + \frac{N\theta}{2} + 1 = -(\alpha+1)\frac{p}{p-m} + 1 + \frac{N\theta}{2}.$$

Then by (4.38) we get from (4.36)

$$(4.39) \quad \int_0^T \int_{\Omega_T} |u|^p \psi(t, x) dt dx \leq CT^\sigma,$$

where

$$\sigma = \max \left( -(\alpha + 2) \frac{p}{p-1} + \frac{N\theta}{2} + 1, -(\alpha + 1) \frac{p}{p-m} + 1 + \frac{N\theta}{2} \right).$$

Then we distinguish two principal cases:

**Case 1:** If  $\sigma \leq 0$

This case itself is divided into two subcases as follows:

**1. i. Subcase of  $\sigma < 0$**

In this case we have  $-(\alpha + 2) \frac{p}{p-1} + \frac{N\theta}{2} + 1 < 0$  and  $-(\alpha + 1) \frac{p}{p-m} + 1 + \frac{N\theta}{2} < 0$ , so the condition  $-(\alpha + 2) \frac{p}{p-1} + \frac{N\theta}{2} + 1 < 0$  implies

$$N < -\frac{2p(\alpha + 1) + 2}{\left( \frac{(\alpha+1)(p-1)}{m-p} + \alpha \right) (p-1)},$$

and the condition  $-(\alpha + 1) \frac{p}{p-m} + 1 + \frac{N\theta}{2} < 0$  implies

$$N < \frac{2(\alpha p + m)}{\alpha(m-1) + (p-1)},$$

where we have replaced  $\theta$  by its value. This means that

$$(4.40) \quad N < \min \left( \frac{2(\alpha p + m)}{\alpha(m-1) + (p-1)}, -\frac{2p(\alpha + 1) + 2}{\left( \frac{(\alpha+1)(p-1)}{m-p} + \alpha \right) (p-1)} \right).$$

If we come back and replace  $\alpha$  by its value  $1 - \gamma$  in (4.40) we get

$$(4.41) \quad N < \min \left( \frac{2((1-\gamma)p + m)}{(1-\gamma)(m-1) + (p-1)}, -\frac{2p(2-\gamma) + 2}{\left( \frac{(2-\gamma)(p-1)}{m-p} + (1-\gamma) \right) (p-1)} \right).$$

Then if the condition (4.40) (or equivalently 4.41) is satisfied, we pass to the limit as  $T \rightarrow +\infty$  in the formula (4.39) and we get

$$\lim_{T \rightarrow +\infty} \int_0^T \int_{\Omega_T} |u|^p \psi(t, x) dt dx = 0.$$

Using the dominated convergence theorem of Lebesgue (Theorem 1.1.4 page 3 in [3]), the continuity of  $u$  with respect to  $t$  and  $x$  and the fact that

$$(4.42) \quad \lim_{T \rightarrow +\infty} \psi(t, x) = 1,$$

we get

$$\int_0^{+\infty} \int_{\mathbb{R}^N} |u|^p dt dx = 0,$$

which implies that  $u \equiv 0$  and this is a contradiction.

**1. ii. Subcase of  $\sigma = 0$ .**

Firstly, taking the limit as  $T \rightarrow \infty$  in (4.39) with the consideration  $\sigma = 0$ , we see that

$$\int_0^{+\infty} \int_{\mathbb{R}^N} |u|^p dt dx < +\infty.$$

This means that  $u \in L^p((0, +\infty); L^p(\mathbb{R}^N))$  and from which we get

$$(4.43) \quad \lim_{R \rightarrow \infty} \int_0^{+\infty} \int_{\Delta_R} |u|^p \psi dt dx = 0,$$

where

$$\Delta_R := \left\{ x \in \mathbb{R}^N : R^\theta < |x|^2 \leq 2R^\theta \right\} \text{ and } \theta \text{ is defined by (4.37).}$$

Now fixing arbitrarily  $R$  in  $]0, T[$  for some  $T > 0$  and taking in this time

$$\varphi_1(x) = \phi \left( \frac{|x|^2}{T^{\frac{\theta}{2}} R^{-\frac{\theta}{2}}} \right),$$

where  $\phi$  is the function defined by (4.6). Using Hölder's inequality

$$\int_X uv d\mu \leq \left( \int_X u^p d\mu \right)^{\frac{1}{p}} \left( \int_X v^q d\mu \right)^{\frac{1}{q}}; \quad u \in L^p(X), \quad v \in L^q(X), \quad p, q > 1, \quad pq = p+q,$$

instead of the Young's one to estimate the integral  $I_2$  in (4.23) on the set

$$\Omega_{TR^{-1}} := \left\{ x \in \mathbb{R}^N : |x|^2 \leq 2T^\theta R^{-\theta} \right\} = \text{supp} \varphi_1,$$

and noting that  $\Delta_{TR^{-1}} \subset \Omega_{TR^{-1}}$  and the support of  $\Delta\varphi_1$  is contained in  $\Delta_{TR^{-1}}$  where

$$\Delta_{TR^{-1}} := \left\{ x \in \mathbb{R}^N : (TR^{-1})^\theta < |x|^2 \leq 2(TR^{-1})^\theta \right\},$$

and  $\theta$  is always given by (4.37), we get

$$(4.44) \quad \begin{aligned} & \int_0^T \int_{\Omega_{TR^{-1}}} |u| \varphi_1^{r-2} \left[ |\Delta\varphi_1|^2 + |\nabla\varphi_1|^2 \right] \left| D_{t|T}^\alpha \varphi_2 \right| dt dx \leq \\ & \left( \int_0^T \int_{\Delta_{TR^{-1}}} |u|^p \psi dt dx \right)^{\frac{1}{p}} \times \\ & \left( \int_0^T \int_{\Delta_{TR^{-1}}} \psi^{\frac{q}{p}} \varphi_1^{r-2q} \left( |\Delta\varphi_1|^q + |\nabla\varphi_1|^{2q} \right) \left| D_{t|T}^\alpha \varphi_2 \right|^q dt dx \right)^{\frac{1}{q}}. \end{aligned}$$



Recalling the integrals  $I_1, I_3$  in page (648) and  $\tilde{I}_2$  such that

$$\tilde{I}_2 := \left( \int_0^T \int_{\Delta_{TR^{-1}}} \psi^{\frac{a}{p}} \varphi_1^{r-2q} \left( |\Delta \varphi_1|^q + |\nabla \varphi_1|^{2q} \right) \left| D_{t|T}^\alpha \varphi_2 \right|^q dt dx \right)^{\frac{1}{q}}.$$

To estimate them, we use the scaled variables  $x = T^{\frac{\theta}{2}} R^{-\frac{\theta}{2}} y$ , and  $t = T\tau$  on the set  $\Omega_{TR^{-1}}$ . We get firstly

$$(4.45) \quad I_1 + I_3 \leq C \left( T^{1-(\alpha+2)\frac{p}{p-1}+N\frac{\theta}{2}} + T^{1-(\alpha+1)\frac{p}{p-m}+N\frac{\theta}{2}} \right) R^{-N\theta},$$

and using the hypothesis  $\sigma = 0$  we obtain from (4.45)

$$(4.46) \quad I_1 + I_3 \leq CR^{-N\theta/2}.$$

Computing the integral  $\tilde{I}_2$  using the same scaled variables and the same form of the function  $\varphi_1$  and using (4.46) we get from (4.23)

$$(4.47) \quad \begin{aligned} \int_0^T \int_{\Omega_{TR^{-1}}} |u|^p \psi dt dx &\leq CR^{-N\theta/2} \\ &+ CR^{\theta-\frac{N\theta}{2q}} \left( \int_0^T \int_{\Delta_{TR^{-1}}} |u|^p \psi dt dx \right)^{\frac{1}{p}}. \end{aligned}$$

Now taking the limit as  $T \rightarrow +\infty$  in (4.47), using (4.43) and (4.42) we get

$$\int_0^\infty \int_{\mathbb{R}^N} |u|^p dt dx \leq CR^{-N\theta/2},$$

which means that necessarily  $R \rightarrow +\infty$ . This contradicts our hypothesis. Noting that condition  $\sigma = 0$  is equivalent to

$$(4.48) \quad \begin{aligned} N &= \frac{2((1-\gamma)p+m)}{(1-\gamma)(m-1)+(p-1)} \quad \text{or} \\ N &= -\frac{2p(2-\gamma)+2}{\left(\frac{(2-\gamma)(p-1)}{m-p}+(1-\gamma)\right)(p-1)}. \end{aligned}$$

Then by (4.41) and (4.48) we have

$$N \leq \min \left( \frac{2((1-\gamma)p+m)}{(1-\gamma)(m-1)+(p-1)}, -\frac{2p(2-\gamma)+2}{\left(\frac{(2-\gamma)(p-1)}{m-p}+(1-\gamma)\right)(p-1)} \right).$$

The second main case is

**Case 2:** if  $p \leq \frac{1}{\gamma}$ .

Even this case is divided into two subcases as follows:

**2. i. Subcase of  $p < \frac{1}{\gamma}$ .**

In this case we recall (4.23) and we take  $\varphi_1(x) = \phi\left(\frac{|x|^2}{R^\theta}\right)$  where  $\phi$  is the function defined by (4.6) and  $R$  is a fixed positive number. Trying to estimate the integrals  $I_1$ ,  $I_2$  and  $I_3$  (page(648)) with respect to  $x$  on the set

$$\Sigma_R = \left\{x \in \mathbb{R}^N : |x| \leq 2R^{\theta/2}\right\} = \text{supp}\varphi_1.$$

Using the scaled variables  $x = R^{\frac{\theta}{2}}y$ ,  $t = T\tau$  for the first integral we find

$$\begin{aligned} & \int_0^T \int_{\Sigma_R} \varphi_1^r \varphi_2^{-\frac{1}{p-1}} \left| D_{t|T}^{\alpha+2} \varphi_2 \right|^{\frac{p}{p-1}} dt dx = \\ & \left( \int_{\Sigma_R} \varphi_1^r dx \right) \left( \int_0^T \varphi_2^{-\frac{1}{p-1}} \left| D_{t|T}^{\alpha+2} \varphi_2 \right|^{\frac{p}{p-1}} dt \right) \\ (4.49) \quad & = \left( R^{N\theta/2} \int_0^1 \phi^r(y^2) dy \right) \times \\ & \left( T^{1-(\alpha+2)\frac{p}{p-1}} \int_0^1 (1-\tau)^{-\frac{\beta}{p-1} + (\beta-\alpha-2)\frac{p}{p-1}} d\tau \right) \\ & = CR^{\frac{N\theta}{2}} T^{1-(\alpha+2)\frac{p}{p-1}}. \end{aligned}$$

In the same way, we have

$$\begin{aligned} & \int_0^T \int_{\Sigma_R} \mu(\varphi_1) \varphi_1^{r-2q} \varphi_2^{-\frac{1}{p-1}} \left| D_{t|T}^\alpha \varphi_2 \right|^q dt dx = \\ (4.50) \quad & \left( \int_{\Sigma_T} \mu(\varphi_1) \varphi_1^{r-2q} dx \right) \left( \int_0^T \varphi_2^{-\frac{1}{p-1}} \left| D_{t|T}^\alpha \varphi_2 \right|^q dt \right) \\ & = CR^{\frac{N\theta}{2}-\theta\frac{p}{p-1}} T^{1-\alpha\frac{p}{p-1}}. \end{aligned}$$

We deduce by using the mean value theorem that

$$\begin{aligned} & \int_0^T \int_{\Sigma_R} \varphi_1^r |\varphi_2|^{-\frac{m}{p-m}} |g| \left| D_{t|T}^{\alpha+1} \varphi_2(t) \right|^{\frac{p}{p-m}} dt dx = \\ (4.51) \quad & \left( \int_{\Sigma_T} |g| \varphi_1^r dx \right) \left( \int_0^T |\varphi_2|^{-\frac{m}{p-m}} \left| D_{t|T}^{\alpha+1} \varphi_2(t) \right|^{\frac{p}{p-m}} dt \right) \\ & = CR^{\frac{N\theta}{2}} T^{1-(\alpha+1)\frac{p}{p-m}}. \end{aligned}$$

Using the formulas (4.49), (4.50) and (4.51) we arrive at

$$\begin{aligned} (4.52) \quad & \int_0^T \int_{\Sigma_R} |u|^p \psi(t, x) dt dx = CR^{\frac{N\theta}{2}} \left( T^{1-(\alpha+2)\frac{p}{p-1}} + T^{1-(\alpha+1)\frac{p}{p-m}} \right) \\ & + CR^{\left(\frac{N}{2}-\frac{p}{p-1}\right)\theta} T^{1-\alpha\frac{p}{p-1}}. \end{aligned}$$

Firstly, passing to the limit in (4.52) as  $T \rightarrow +\infty$  and using the fact that  $\lim_{T \rightarrow +\infty} \psi(t, x) = 1$ , we get

$$(4.53) \quad \int_0^{+\infty} \int_{\Sigma_R} |u|^p dt dx = 0,$$

Secondly, taking the limit in (4.53) as  $R \rightarrow +\infty$  we obtain

$$\int_0^{+\infty} \int_{\mathbb{R}^N} |u|^p dt dx = 0, \quad \text{whereupon } u \equiv 0.$$

which is a contradiction.

## 2. ii. Subcase of $p = \frac{1}{\gamma}$

In this case we further assume that

$$(4.54) \quad \frac{N}{2} - \frac{p}{p-1} < 0,$$

which is equivalent to  $\alpha < \frac{2}{N}$  or  $\frac{N-2}{N} < \gamma < 1$  since  $\alpha = 1 - \gamma$ . Under this assumption, we have

$$(4.55) \quad \begin{aligned} 1 - (\alpha + 2) \frac{p}{p-1} &= -\frac{2}{\alpha} < 0; \quad 1 - \alpha \frac{p}{p-1} = 0; \\ 1 - (\alpha + 1) \frac{p}{p-m} &= \frac{m\alpha - m - \alpha}{m\alpha - m + 1} < 0. \end{aligned}$$

Hence, passing to the limit as  $T \rightarrow \infty$  in (4.52) and the fact that (4.55) and (4.42) are fulfilled, we obtain

$$(4.56) \quad \int_0^\infty \int_{\Sigma_R} |u|^p dt dx = CR^{(\frac{N}{2} - \frac{p}{p-1})\theta}.$$

Finally, passing to the limit as  $R \rightarrow \infty$  in (4.56), using the condition (4.54) and the fact that  $\theta > 0$ , we get

$$\int_0^\infty \int_{\mathbb{R}^N} |u|^p dt dx = 0, \quad \text{whereupon } u \equiv 0.$$

This is exactly the desired contradiction. The proof of *Theorem 4.1* is achieved.

## 5. Acknowledgements

The authors would like to express their sincere thanks to the reviewers for their valuable suggestions. The authors are also thankful to the editor-in-chief for his important comments which helped to improve the presentation of the paper.

## REFERENCES

1. R. ADAMS: *Sobolev spaces*. Academic Press, New York-London, 1975.
2. HAIM BRÉZIS: *Analyse Fonctionnelle, Théorie et applications*. Dunod, Paris, 1999.
3. VIOREL BARBU: *Partial Differential Equations and Boundry Value Problems*. Science +Business media, B.V. Springer. Vol 441.
4. M.E. TAYLOR: *Partial differential equations III, Nonlinear Equations*. Springer, New York, 1996.
5. H. FUJITA: *On the Blowing up of solutions of the problem for  $u_t = \Delta u + u^{1+\alpha}$* . Faculty of science, University of Tokyo13 (1966), 109-124.
6. T.CAZENAVE, F. DICKSTEIN and F. D. WEISSLER: *An equation whose Fujita critical exponent is not given by scaling*, Nonlinear anal. **68**(2008), 862-874.
7. A. FINO: *Critical exponent for damped wave equations with nonlinear memory*. Hal Arch. Ouv. Id. 00473941v2, (2010).
8. M. BERBICHE and A. HAKEM: *Finite time blow-up of solutions for damped wave Equation with non linear Memory*. Comm. Math. Analysis. **(14)(1)**(2013), 72-84.
9. S. SELBERG: *Lecture Notes*. Math. 632, PDE, <http://www.math.ntnu.no/ssselberg>, 2001.
10. IGORE PODLUBNY: *Fractional Differetial Equations*. Mathematics in science and engineering. Volume 198, University of Kosice, Slovak republic.
11. G. TODOROVA and B. YARDANOV: *Critical exponent for a non linear wave equation with damping*. Journal of Differential equations. **(174)** (2001), 464-489.
12. QI S. ZHANG: *A Blow up result for a nonlinear wave equation with damping*. C.R. Acad. Sciences, Paris, 2001.
13. S. KATAYAMA, MD A. SHEIKH and S. TARAMA: *The Cauchy and mixed problems for semilinear wave equations with damping terms*. Math. Japonica. **50 (3)**(2000), 459-566.
14. MD. ABU NAIM SHEIKH and MD. ABDUL MATIN: *Global Existence of Solution for semilinear Dissipative wave Equation*. Vietnam Journal of math. **34:3**(2006), 295-305.
15. S. I. POHOZAEV and A. TESEI: *Blow-up of nonnegative solutions to quasilinear parabolic inequalities*. Atti Accad. Naz. Lincei Cl. Sci. Fis. Math. Natur. Rend. Lincei. 9 Math. App. 11, N2, (2000), 99-109.
16. E. MITIDIERI and S.I. POHOZAEV: *Nonexistence of weak solutions for some degenerate elliptic and parabolic problems on  $\mathbb{R}^N$* . J. Evol. Equations. (2001), 189-220.
17. E. MITIDIERI and S.I. POHOZAEV: *A priori estimates and blow-up of solutions to nonlinear partial differential equations and inequalities*. Proc. Steklov. Inst. Math. **(234)**(2001), 1-383.
18. S. G. SAMKO, A. A. KILBAS and O. I. MARICHEV: *Fractional Integrals and derivatives, Theory and application*. Gordon and Breach Publishers. (1987).
19. J. L. LIONS and W. A. STRAUSS: *Some nonlinear evolution equations*. Bull. Soc. Math. France.**(93)**(1965), 43-96.

20. H. FAOUR, A. FINO and M. JAZAR: *Local existence and uniqueness for a semilinear accretive wave equation*. J. Math. Anal. Appl. **(377)**(2011), 534–539.
21. P. SOUPLET: *Monotonicity of solutions and blow-up for semilinear parabolic equations with nonlinear memory*. Z. angew. Math. Phys. **55**(2004), 28–31.

Hadj Kaddour Tayeb  
Laboratory ACEDP  
Djillali Liabes university  
22000 Sidi Bel Abbès, Algeria  
`hktn2000@yahoo.fr`

Hakem Ali  
Laboratory ACEDP  
Djillali Liabes university  
22000 Sidi Bel Abbès, Algeria  
`hakemali@yahoo.fr`